Auffrischungskurs Mathematik

– Lösungen zur Selbstkontrolle –

WS 2022/23

Thema 6

Aufgabe 1: *Additionstheoreme*

(a)
$$\begin{aligned}\cos(x \pm y) &= \sin\left(x \pm y + \frac{\pi}{2}\right) = \sin(x \pm z^{\pm}), \qquad z^{\pm} \equiv y \pm \frac{\pi}{2}\\ &= \sin(x)\cos(z^{\pm}) \pm \cos(x)\sin(z^{\pm})\\ &= \sin(x)\underbrace{\cos\left(y \pm \frac{\pi}{2}\right)}_{\mp\sin(y)} \pm \cos(x)\underbrace{\sin\left(y \pm \frac{\pi}{2}\right)}_{\pm\cos(y)}\\ &= \cos(x)\cos(y) \mp \sin(x)\sin(y)\end{aligned}$$

(b)
$$\begin{aligned}\tan(x \pm y) &= \frac{\sin(x \pm y)}{\cos(x \pm y)} = \frac{\sin(x)\cos(y) \pm \cos(x)\sin(y)}{\cos(x)\cos(y) \mp \sin(x)\sin(y)} = \frac{\cancel{\cos(x)\cos(y)}\left(\frac{\sin(x)}{\cos(x)} \pm \frac{\sin(y)}{\cos(y)}\right)}{\cancel{\cos(x)\cos(y)}\left(1 \mp \frac{\sin(x)\sin(y)}{\cos(x)\cos(y)}\right)}\\ &= \frac{\tan(x) \pm \tan(y)}{1 \mp \tan(x)\tan(y)}\end{aligned}$$

(c)
- $\sin(2x) = \sin(x + x) = \sin(x)\cos(x) + \cos(x)\sin(x) = 2\sin(x)\cos(x)$
- $\cos(2x) = \cos(x + x) = \cos^2(x) - \underbrace{\sin^2(x)}_{1-\cos^2(x)} = 2\cos^2(x) - 1$

Aufgabe 2: *Trigonometrische Umformungen I*

(a) $$\tan\left(\frac{\pi}{4} + \alpha\right) = \frac{\sin\left(\frac{\pi}{4} + \alpha\right)}{\cos\left(\frac{\pi}{4} + \alpha\right)} = \frac{\cos\frac{\pi}{4}\sin\alpha + \sin\frac{\pi}{4}\cos\alpha}{\cos\frac{\pi}{4}\cos\alpha - \sin\frac{\pi}{4}\sin\alpha} = \frac{\cancel{\sqrt{2}}\cos\alpha + \sin\alpha}{\cancel{\sqrt{2}}\cos\alpha - \sin\alpha} \qquad \square$$

(b) $$\frac{1 + \sin 2\alpha}{\cos 2\alpha} = \frac{1 + 2\sin\alpha\cos\alpha}{\cos^2\alpha - \sin^2\alpha} = \frac{(\cos\alpha + \sin\alpha)^{\cancel{2}}}{\cancel{(\cos\alpha + \sin\alpha)}(\cos\alpha - \sin\alpha)} \qquad \square$$

(c) $$\begin{aligned}&2\cos\left(\tfrac{x+y}{2}\right)\cos\left(\tfrac{x-y}{2}\right) = 2\left(\cos\tfrac{x}{2}\cos\tfrac{y}{2} - \sin\tfrac{x}{2}\sin\tfrac{y}{2}\right)\left(\cos\tfrac{x}{2}\cos\tfrac{y}{2} + \sin\tfrac{x}{2}\sin\tfrac{y}{2}\right)\\ &= 2\left(\cos^2\tfrac{x}{2}\cos^2\tfrac{y}{2} - \sin^2\tfrac{x}{2}\sin^2\tfrac{y}{2}\right) = 2\left(\cos^2\tfrac{x}{2} + \cos^2\tfrac{y}{2} - 1\right) = \cos(x) + \cos(y) \qquad \square\end{aligned}$$

(d) $$\begin{aligned}&\cot\alpha\cot\beta + \cot\alpha\cot\gamma + \cot\beta\cot\gamma = \frac{\overbrace{\cos\alpha\cos\beta\sin\gamma + \cos\alpha\cos\gamma\sin\beta}^{\cos\alpha\sin(\beta+\gamma)} + \cos\beta\cos\gamma\sin\alpha}{\sin\alpha\sin\beta\sin\gamma}\\ &= \frac{\cos\alpha\sin(\beta+\gamma) + \sin\alpha[\cos(\beta+\gamma) + \sin\beta\sin\gamma]}{\sin\alpha\sin\beta\sin\gamma} = \frac{\sin(\alpha+\beta+\gamma)}{\sin\alpha\sin\beta\sin\gamma} + 1 = 1 \qquad \square\end{aligned}$$

Aufgabe 3: *Trigonometrische Umformungen II*

(a) $1 + \cos\alpha + \cos\frac{\alpha}{2} = 4\cos\frac{\alpha}{2}\cos\left(\frac{\alpha}{4} + \frac{\pi}{6}\right)\cos\left(\frac{\alpha}{4} - \frac{\pi}{6}\right)$

(b) $\frac{2\sin\beta - \sin(2\beta)}{2\sin\beta + 2\sin(2\beta)} = \frac{\sin^2(\beta/2)}{2\cos(\beta/2 + \pi/6)\cos(\beta/2 - \pi/6)}$

(c) $\sin\alpha + \sin\beta + \sin\gamma = 4\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\sin\left(\frac{\alpha}{2} + \frac{\beta}{2}\right)$

Aufgabe 4: *Goniometrische Gleichungen und Gleichungssysteme*

(a) $x_1 = 2\pi k,\ x_2 = \frac{\pi}{2} + 2\pi k$ mit $k \in \mathbb{Z}$

(b) $\cos x = \cos y = \frac{1}{2}$

(c) $\sin(x_1) = 1$ und $\sin(x_{2/3}) = \frac{-1 \pm \sqrt{5}}{4}$

(d) $a + b \neq 0$: $\tan(x_1) = 1$ und $\tan(x_2) = -\frac{1}{2}$
$a + b = 0$: $x \in \mathbb{R}$

Aufgabe 5: *Dreiecksfläche*

$$A = \frac{w(a+b)}{4ab}\sqrt{4a^2b^2 - w^2(a+b)^2}$$

Aufgabe 6: *Sehnen im Kreis*

$$A = \frac{ab}{4r}\left(a\sqrt{1 - \frac{b^2}{4r^2}} + b\sqrt{1 - \frac{a^2}{4r^2}}\right)$$